and

$$d_1(S,K,T) = \frac{\ln(S/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2(S,K,T) = d_1(S,K,T) - \sigma\sqrt{T}.$$

For simplicity we write $c(S, K, T)$ rather than $c(S, K, T, r, \sigma)$ etc. We denote by T the option maturity, and by K – the strike price; the current time is 0. We skip the time subscript if $t = 0$, so $S = S_0$ etc.
If $\tau < T$ then an option written on the dividend paying stock $\hat{S}_t$ has the same value as if it were written on S_t. But it is $\hat{S}$ rather than S we want to see in option pricing formulae, since $\hat{S}$ is the market price of the underlying stock at time 0.
Since $S = \hat{S} - De^{-r\tau}$, the values of options written on a dividend-paying stock are as follows:

(1)
$$c(\hat{S},K,T,D,\tau) = c(\hat{S} - De^{-r\tau}, K, T),$$
$$p(\hat{S},K,T,D,\tau) = p(\hat{S} - De^{-r\tau}, K, T).$$

Pricing dividend-protected options in this model is not difficult either: it is enough to substitute $K - D$ for the strike price K in (1).

3. WSE stock options

Problems arise, however, when we consider WSE stock options. The modification of the option strike price is contingent on the underlying stock price on the last cum-dividend day τ. On the next trading day a WSE option becomes a standard option with strike K if $\hat{S}_\tau > 10D$ *(small dividend)*, or $K - D$ if $\hat{S}_\tau < 10D$ *(large dividend)*.
It is clear that a WSE stock option is equivalent to a portfolio of two *binary* compound options. We shall use this fact to write down formulae for WSE option prices. Of course the formulae can be derived directly; for example one can use the exotic option pricing method in Skipper and Buchen (2003).
The following two sections are devoted to compound options and pricing formulae in the standard Black-Scholes setup, where the underlying asset *St* is a stock that pays no dividends.

3.1. **Compound options.** These are options written on options; so there are four possibilities: one may write calls on calls, calls on puts, puts on calls, and puts on puts. The holder of a compound call (put) has a right to buy (sell) at time τ the underlying vanilla option for κ (compound option strike price). The underlying option has maturity $T > \tau$ and strike K.

The price formulae in the Black-Scholes environment are due to Geske (1979). Details can be found e.g. in Hull (2002). The type of the underlying option will be marked by a subscript, so c_p is the price of a call written on a put etc. Then

$$\begin{aligned}
c_c &= SN_2(a_1, b_1, \rho) - Ke^{-rT}N_2(a_2, b_2, \rho) - \kappa e^{-r\tau}N(a_2),\\
c_p &= Ke^{-rT}N_2(-a_2, -b_2, \rho) - SN_2(-a_1, -b_1, \rho) - \kappa e^{-r\tau}N(-a_2),\\
p_c &= Ke^{-rT}N_2(-a_2, b_2, -\rho) - SN_2(-a_1, b_1, -\rho) + \kappa e^{-r\tau}N(-a_2),\\
p_p &= SN_2(a_1, -b_1, -\rho) - Ke^{-rT}N_2(a_2, -b_2, -\rho) + \kappa e^{-r\tau}N(a_2).
\end{aligned}$$

Here $\rho = \sqrt{(\tau/T)}$, and $N_2(x_1, x_2, \rho)$ is the cumulative bivariate normal distribution with correlation coefficient ρ. The parameters a_1, a_2, b_1, b_2 correspond to d_1 and d_2 in the standard Black-Scholes formulae:

(2)
$$a_i = d_i(S, S^*, \tau), \qquad b_i = d_i(S, K, T), \qquad \text{for } i = 1, 2.$$

*S** is the so-called *critical asset price.* This is the stock price for which the underlying option is worth κ at time τ: $c(S^*, K, T - \tau) = \kappa$, if the underlying option is a call, and $p(S^*, K, T - \tau) = \kappa$, if it is a put. Thus, the compound option holder will exercise it if $S_\tau > S^*$, provided that the option he holds is a call on a call, or a put on a put. For calls on puts, and puts on calls, the condition is $S_\tau < S^*$.

3.2. **Binary compound options.** The holder of a binary compound call receives the underlying vanilla option *for free,* provided that on maturity τ it is worth at least κ. A standard compound call is then equivalent to a portfolio consisting of a long binary compound call and a short cash or nothing call, so the formula for its value follows immediately from the price formula for the standard compound call:

$$\begin{aligned}
bc_c &= SN_2(a_1, b_1, \rho) - Ke^{-rT}N_2(a_2, b_2, \rho),\\
bc_p &= Ke^{-rT}N_2(-a_2, -b_2, \rho) - SN_2(-a_1, -b_1, \rho).
\end{aligned}$$

In analogy to standard binary asset or nothing puts, the holder of a binary compound put *receives* the underlying option for free if at time τ its value is *at most* κ. This leads to a different relationship than the one above: A standard compound put is equivalent to a portfolio containing a *short* binary compound put and a *long* cash or nothing put. In effect

$$\begin{aligned}
bp_c &= SN_2(-a_1, b_1, -\rho) - Ke^{-rT}N_2(-a_2, b_2, -\rho),\\
bp_p &= Ke^{-rT}N_2(a_2, -b_2, -\rho) - SN_2(a_1, -b_1, -\rho).
\end{aligned}$$

3.3. **Pricing WSE calls and puts.** We assume that the nearest dividend has already been voted, so its value D and the last cum-dividend date τ are known with certainty.

A WSE call option is equivalent to a portfolio consisting of two long binary compound options that mature at time τ: a call written on a call$_K$, and a put written on a call$_{K-D}$; the subscripts indicate the value of the option strike price.

Note that we do not need the strike prices of binary compound options; it is sufficient to know the critical asset price. This is convenient because the compound options in our portfolio have different strike prices, but the same critical asset price $\hat{S}^* = 10D$.

We assume the notation as in Section 2. Recall that $\hat{S}_\tau = S_\tau + D$. If we consider WSE options as written on the risky component S_t of $\hat{S}_t$, then the critical price $S^* = 9D$. Note also that at time $t = 0$ we have $S = \hat{S} - De^{-r\tau}$. In effect, we have the following formula for the price of WSE calls:

$$
\begin{aligned}
c_{\text{wse}} &= (\hat{S} - De^{-r\tau})N_2(a_1, b_1, \rho) - Ke^{-rT}N_2(a_2, b_2, \rho) \\
&\quad + (\hat{S} - De^{-r\tau})N_2(-a_1, b_1', -\rho) - (K - D)e^{-rT}N_2(-a_2, b_2', -\rho).
\end{aligned}
$$

Similarly, a WSE put is a portfolio consisting of a long put on a put (K) and a long call on a put ($K — D$), so

$$
\begin{aligned}
p_{\text{wse}} &= Ke^{-rT}N_2(a_2, -b_2, -\rho) - (\hat{S} - De^{-r\tau})N_2(a_1, -b_1, -\rho) \\
&\quad + (K - D)e^{-rT}N_2(-a_2, -b_2', \rho) - (\hat{S} - De^{-r\tau})N_2(-a_1, -b_1', \rho).
\end{aligned}
$$

Here

$$
\begin{aligned}
a_i &= d_i(\hat{S} - De^{-r\tau}, 9D, \tau), \qquad & b_i &= d_i(\hat{S} - De^{-r\tau}, K, T), \\
b_i' &= d_i(\hat{S} - De^{-r\tau}, K - D, T), \qquad & &\text{for } i = 1 \ldots 2.
\end{aligned}
$$

Note that if we set $D = 0$ then $b_i = b'_i$ and the standard Black-Scholes equations can be recovered by using the identity

$$
N_2(a, b, \rho) + N_2(-a, b, -\rho) = N(b).
$$

3.4. **Numerical calculations.** It is not difficult to implement the above formulae in order to calculate numeric results (see Footnote 6). One needs, however, an implementation of the bivariate cumulative normal distribution. This, in turn, relies on the univariate cumulative normal distribution. A very interesting discussion of the accuracy issues that arise, and implementations of the distributions can be found in West (2004).

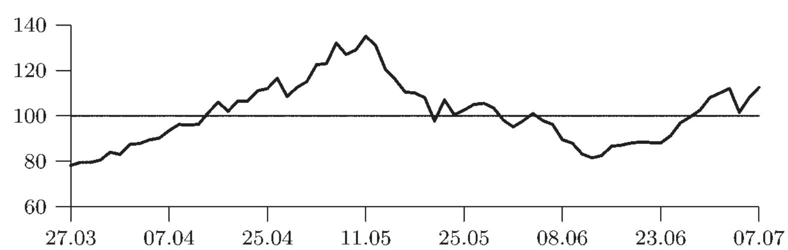
3.5. **Pricing errors.** The official documents that specify margin requirements for the derivatives traded on the Warsaw Stock Exchange contain Black-Scholes formulae in the version where the spot price is adjusted for dividends, as in Formula (1). This is sufficient in most cases. After all, dividends are usually *small.*
However, there was an exception in the past when the dividends were high enough to affect option prices significantly. Table 1 shows the dividends per share paid by KGHM in the last three years.[5]

Table 1. KGHM dividends

Last cum dividend day	Stock price (close)	Dividend value
5 July 2005	34.10	2.00
4 July 2006	112.00	10.00
20 June 2007	124.00	16.97

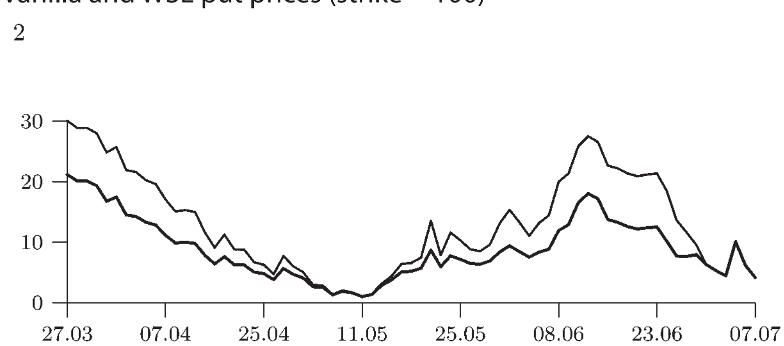
The relevant year is 2006, in the period when September and December options were traded. Graph 2 shows KGHM stock prices between 27.03.2006 and 7.07.2006. The prices kept oscillating around the critical value $\hat{S}^* = 100$, and it was uncertain till the last moment whether KGHM options would be modified.

Graph 2: KGHM stock prices (close) in 2006



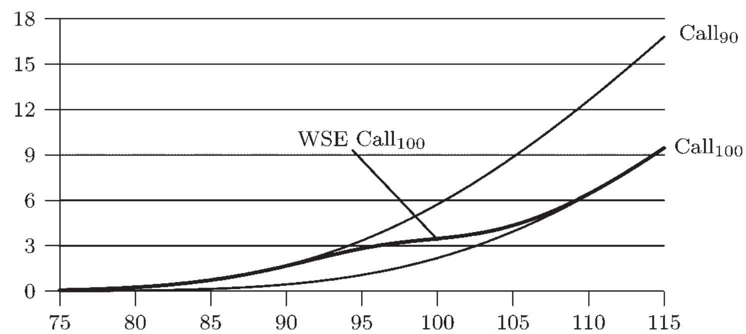
The theoretical[6] vanilla and WSE put prices are shown in Graph 3. In some periods the difference between the prices was significant, with the vanilla put price exceeding the WSE put price by more than 50%. Similar plots can be made for call options.[7]

Graph 3: Vanilla and WSE put prices (strike = 100)



Let us fix the time to maturity and see how vanilla and WSE option prices depend on the stock price. Graph 4 shows the prices of WSE call_{100}, vanilla call_{100} and vanilla call_{90} as a function of the underlying stock price (calculated 9 days before the last cum-dividend day). It is clear that WSE call prices converge to call_{90} or call_{100} prices depending on the stock price level, or — on the probability of the WSE option being modified.

Graph 4: Vanilla and WSE call prices as functions of S



4. Put-Call parity

The standard put-call parity relationship relies on the fact that the portfolio consisting of a long call_K and a short put_K replicates a forward contract with delivery price K:

(3) $$c - p = \hat{S} - De^{-r\tau} - Ke^{-rT}.$$

4.1. **Put-call parity for WSE stock options.** In this case Formula (3) is no longer valid: if the dividend turns out *large*, then the strike prices are modified, and the long call-short put portfolio turns into a long forward with delivery price $K - D$. Note that in this case the value of both components of the portfolio increases. The total gain at time T is equal to D, provided that $S_\tau \leq 9D$. Consequently, the put-call parity formula should be adjusted by the value of a binary cash or nothing put:

$$c_{\text{wse}} - p_{\text{wse}} = \hat{S} - De^{-r\tau} - Ke^{-rT} + De^{-rT}N(-a_2).$$

This means that (if there are dividends) there is no simple way of replicating WSE calls with futures and puts, nor – to replicate puts with futures and calls. This hurts liquidity: with large bid-ask spreads (the market-makers operating on WSE are obliged to keep bid-ask spreads at 20% or less) it was sometimes advantageous to close a position in a call option with a put and a futures contract etc. With large dividends around, this could generate additional risks. The same problem appears in put-call parity arbitrage trades.

4.2. **Put-Call parity and "risky arbitrage"**. Trading in WSE stock options has never been particularly active, and ocassional arbitrage opportunities did occur. The most common case was when the put-call parity relationship was not satisfied.
Consider the put-call parity formula for vanilla options, written in terms of the futures price F:

$$c - p = (F - K)e^{-rT}.$$

There are two possibilities. If $c - p < (F - K)e^{-rT}$ then the arbitrageur sells a futures contract, buys a call_K and writes a put_K. If the opposite inequality holds then the arbitrageur buys a futures contract, writes a call_K and buys a put_K. These transactions work fine for vanilla options (e.g. the index options listed on the WSE), or when no dividends are paid during the lifetime of a WSE stock option.
If a dividend D is paid at time $\tau < T$, then the first "arbitrage" position in WSE stock options is in fact a long binary cash or nothing put, so it will generate extra profit D at time T, if $\hat{S}_\tau \leq 10D$. On the other hand the "arbitrage" position of the second type is a short cash or nothing put, and the investor risks that the strike price would be modified.

Example. The following table presents one of the arbitrage trades carried out by the Author on 3.01.2006, at 12.45:

Transaction	Security	Price	Position
buy	FPEOM6	181.0	Long futures
buy	OPEOR6180	13.2	Long put
write	OPEOF6180	19.0	Short call

Each option/contract is written on 100 PKO S.A. stocks, strike price 180, maturity 16 June 2006. The put-call parity formula is not satisfied and the transactions generated profit (with commissions and opportunity costs taken into account). But the position, being a short binary cash or nothing put, was not riskless.

To make the transaction a proper arbitrage, one could try to delta hedge the position. Delta hedging binary options can prove difficult (in some cases the option delta can get out of control), but this case was worse: In 2006 the last cum-dividend date was 19.05.2006. The dividend of 7.40 was voted on 04.05.2006 — so its value was not known at the time when the "arbitrage" transactions were carried out. For this reason it was neither possible to price the position correctly, nor to delta hedge it.
In the years 2002-2005 the dividend to the stock price ratio (on the close of the last cum-dividend day) ranged between 3.4 to 5.2 per cent, well below the 10% limit, so it seemed likely that the PKO S.A. options would not be modified (this indeed turned out to be the case). But the unexpected may happen. The additional risk factor, due to the uncertain dividend value, is best illustrated by the KGHM dividends paid in recent years.

4.3. **The KGHM** case. The dividend paid by KGHM in 2005 was just PLN 2 per share (see Table 3.5), but by July 2006 the KGHM stock price more than tripled, one of the reasons being the increase in the price of copper.
So was it possible to predict the value of the dividend paid in 2006? The knowledge of the 2005 profit would not have helped much: the KGHM Management Board recommended (23.03.06) in the 2005 annual report that a dividend of 3.50 per share be paid. This was later (26.04.06) adjusted to 5.50. Eventually (14.06.06) the main shareholder (the Polish Treasury, which owns just over 40% of KGHM stocks) managed to wrestle a PLN 10 dividend (87.36% of the 2005 profit).
In 2007 a PLN 7.0 dividend was recommended (20.03.07) but the Treasury had its way again and on 30.05.07 a dividend of 16.98 was voted. This was supposed to constitute 100% of the 2006 profit but some time later it was realised that the dividend figure was wrong: the PLN 16.97565 profit per share (approx.) had been rounded up. In effect the actual dividend payment would exceed the company's profit by some PLN 870,000! A new meeting was called on 9.07.07 (after the last cum-dividend day!) and the dividend was adjusted to 16.97. It was to be paid in two installments (in July and in September 2007).
If nothing else, shareholders do not go for large dividends for tax reasons. But in this case the main shareholder, being the State Treasury, not only pays no tax, but it receives tax payments from the other shareholders. No surprise that not all were entirely happy. The decision of the shareholders' meeting has been challenged in court by a shareholder who owns just one KGHM stock. He was unhappy that the company was completely stripped of its earnings. In September 2007 the court blocked the payment of the second part of the dividend[8].
In the meantime the KGHM stock price stayed well below the massive critical price $\hat{S}^* = 169.7$, and in fact all KGHM options were modified on 26 June 2007: the strike prices were reduced by 16.97 and rounded to the nearest integer. The WSE adjusted

the option codes and stockbrookers had to follow suit by reprogramming their trading software. For example OKGHU7120 turned into OKGHU7103D and so on. All this in vain: so far (October 2007), not a single September nor a December stock option has ever been traded!

Interestingly, this was the only case when WSE stock options were modified. What is more, it coincided with the first case in the WSE history that a dividend payment was challenged in court, and blocked. This shows that the additional risk factor of uncertain dividends can exceed the risks that arise from inappropriate pricing models.

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(Footnotes)

1 KGHM, PKN Orlen, PKO S.A., Prokom, and TP S.A.

2 WIG20 index options appeared on the Warsaw Exchange in September 2003.

3 More precisely, what matters here is the arithmetic average of the transaction prices, weighted by the respective trading volumes, on the last cum-dividend day. The settlement price on maturity is calculated in the same way. In this paper we simplify things by considering spot prices on the close of the respective trading days, rather than the averages.

4 Warsaw Stock Exchange.

5 The information about the dividends paid by companies listed on the Warsaw Stock Exchange can be found in http://mojeinwestycje.interia.pl.

6 In the numeric calculations $\sigma = 32\%$, and $r = 4.16\%$.

7 An Excel file with the calculatons for this paper can be downloaded from http://www.wsb-nlu.edu.pl/~kobak.

8 Added in print: KGHM appealed against the court decision and won the case (28 November 2007). The second dividend installment was finally paid out on 12 December 2007.